

# Ordering of bicyclic graphs by matching energy \*

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## Abstract

Let  $G$  be a simple graph of order  $n$  and  $\mu_1, \mu_2, \dots, \mu_n$  be the roots of its matching polynomial. The matching energy is defined as the sum  $\sum_{i=1}^n |\mu_i|$ , which was introduced by Gutman and Wagner in 2012. In this paper, the graphs with the first five smallest matching energies among all bicyclic graphs for order  $n > 5$  are determined.

**Key Words:** Bicyclic graph, Matching polynomial, Matching energy.

**AMS Subject Classification (2010):** 05C35, 05C50.

## 1 Introduction

Let  $G = (V, E)$  be a finite, connected, undirected and simple graph with vertex set  $V = V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E = E(G) = \{e_1, e_2, \dots, e_m\}$ . A matching in a graph  $G$  is a set of pairwise nonadjacent edges. A matching  $M$  is called  $k$ -matching if the size of  $M$  is  $k$ . Let  $m(G, k)$  denote the number of  $k$ -matching of  $G$ , where  $m(G, 1) = m$ , and  $m(G, k) = 0$  for  $k > \lfloor \frac{n}{2} \rfloor$  or  $k < 0$ . In addition, we define  $m(G, 0) = 1$ . The matching polynomial of the graph  $G$  is defined as

$$\alpha(G) = \alpha(G, x) = \sum_{k \geq 0} (-1)^k m(G, k) x^{n-2k}.$$

Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of the adjacency matrix of a graph  $G$ . The energy of graph  $G$  [6] is defined as

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

An important tool of graph energy is the Coulson integral formula [6] (with regard to  $G$  being a tree  $T$ ):

$$E(T) = \frac{2}{\pi} \int_0^\infty \frac{1}{x^2} \ln \left[ \sum_{k \geq 0} m(T, k) x^{2k} \right] dx. \quad (1)$$

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\*Supported by the National Natural Science Foundation of China (Nos. 11171273 and 11601431) and the Seed Foundation of Innovation and Creation for Graduate Students in Northwestern Polytechnical University (No. Z2017190).

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The energy of graphs has been widely studied by theoretical chemists and mathematicians. For details see the new book on graph energy [9, 14] and the reviews [7, 10].

Recently, Gutman and Wagner [11] defined the matching energy of a graph  $G$ . Let  $G$  be a simple graph and  $\mu_1, \mu_2, \dots, \mu_n$  be the zeros of its matching polynomial. Then

$$ME(G) = \sum_{i=1}^n |\mu_i|.$$

In view of Eq. (1), the matching energy also has a beautiful formula as follows [11]:

$$ME(G) = \frac{2}{\pi} \int_0^\infty \frac{1}{x^2} \ln \left[ \sum_{k \geq 0} m(G, k) x^{2k} \right] dx. \quad (2)$$

By Eq. (2), we know that the matching energy of a graph  $G$  is a monotonically increasing function of  $m(G, k)$ . This means that if two graphs  $G_1$  and  $G_2$  satisfy  $m(G_1, k) \leq m(G_2, k)$  for all  $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ , then  $ME(G_1) \leq ME(G_2)$ . In addition, if  $m(G_1, k) < m(G_2, k)$  for at least one  $k$ , then  $ME(G_1) < ME(G_2)$ .

We now introduce some elementary notations and terminologies that will be used in the sequel. Let  $G = (V, E)$  be a graph under our consideration. If  $W \subseteq V(G)$ , we denote by  $G - W$  the subgraph of  $G$  obtained by deleting the vertices of  $W$  and the edges incident with them. Similarly, if  $E' \subseteq E(G)$ , we denote by  $G - E'$  the subgraph of  $G$  obtained by deleting the edges of  $E'$ . If  $W = \{v\}$  and  $E' = \{xy\}$ , we write  $G - v$  and  $G - xy$  instead of  $G - \{v\}$  and  $G - \{xy\}$ , respectively. We will use the notations  $S_n$ ,  $P_n$  and  $C_n$  to denote the star, path and cycle on  $n$  vertices, respectively. The union of the graphs  $G_1$  and  $G_2$ , denoted by  $G_1 \cup G_2$ , is the graph with vertex set  $V(G_1) \cup V(G_2)$  and edge set  $E(G_1) \cup E(G_2)$ . Let  $G_1$  and  $G_2$  be two graphs with  $V(G_1) \cap V(G_2) = \{v\}$ . Let  $G = G_1 v G_2$  be a graph defined by  $V(G) = V(G_1) \cup V(G_2)$  and  $E(G) = E(G_1) \cup E(G_2)$ . Let  $u$  be a vertex of  $G$ ,  $N(u)$  or  $N_G(u)$  denotes the neighborhood of  $u$ . We refer to Cvetković et al. [3] for undefined notations and terminologies.

A bicyclic graph is a connected graph with  $n$  vertices and  $n + 1$  edges. Let  $\mathcal{B}(n)$  be the class of bicyclic graphs with  $n$  vertices. We shall use  $B_{n,a,b}^{(t)}$  to denote the bicyclic graph constructed by attaching  $t$  pendent vertices to the vertex  $v$  of  $C_a v C_b$ . Let  $B_{n,a,b}'^{(t)}$  denote the bicyclic graph that is obtained by attaching  $t$  pendent vertices to one vertex except  $v$  of  $C_a v C_b$ . Three internal disjoint paths  $P_x$ ,  $P_y$  and  $P_c$  possessing common end vertices  $u$ ,  $v$  form a bicyclic graph denoted by  $B_{x,y,c}$ . We shall use  $B_{n,x,y,c}^{(t)}$  to denote the bicyclic graph constructed by attaching  $t$  pendent vertices to the vertex  $v$  of  $B_{x,y,c}$ . Let  $B_{n,x,y,c}'^{(t)}$  denote the bicyclic graph that is obtained by attaching  $t$  pendent vertices to one vertex except  $v$  and  $u$  of  $B_{x,y,c}$ . Bicyclic graphs  $B_{n,a,b}^{(t)}$ ,  $B_{n,a,b}'^{(t)}$ ,  $B_{n,x,y,c}^{(t)}$  and  $B_{n,x,y,c}'^{(t)}$  are depicted in Figure 1.

The study on extremal matching energy is interesting. In [11], Gutman and Wagner gave some elementary results on the matching energy and obtained the unicyclic graphs with the minimal and maximal matching energy. For the bicyclic graphs, Ji et al. [12] obtained the graphs with the minimal and maximal matching energy. In [13], Ji and Ma obtained tricyclic graph with maximum matching energy. In [18], Zou and Li characterized the bicyclic graph with given girth having minimum matching energy. For more results about matching energy, see [1, 2, 15, 16, 17].

This paper is organized as follows: In Section 2, we give some preliminary results, which will be used in the following discussion. For  $n > 5$ , the graphs with the first five smallest matching energies among all bicyclic graphs of order  $n$  are determined in Section 3.

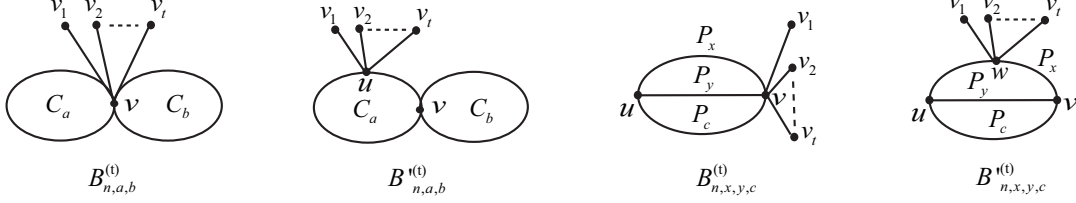


Figure 1: Bicyclic graphs  $B_{n,a,b}^{(t)}$ ,  $B_{n,a,b}'^{(t)}$ ,  $B_{n,x,y,c}^{(t)}$  and  $B_{n,x,y,c}'^{(t)}$ .

## 2 Preliminary Results

In this section, we shall give some elementary results which will be used in the following.

**Lemma 2.1.** ([4, 8]) Let  $G = (V, E)$  be a graph.

- (i) If  $uv \in E(G)$ , then  $m(G, k) = m(G - uv, k) + m(G - u - v, k - 1)$ ;
- (ii) If  $u \in V(G)$ , then  $m(G, k) = m(G - u, k) + \sum_{v \in N(u)} m(G - u - v, k - 1)$ .

**Lemma 2.2.** ([5]) If  $m(G_1, k) \geq m(G_2, k)$ , then  $m(G_1 \cup H, k) \geq m(G_2 \cup H, k)$ , where  $H$  is an arbitrary graph.

**Lemma 2.3.** ([1]) Let  $G$  be a simple graph and  $H$  be a subgraph (resp. proper subgraph) of  $G$ . Then  $m(G, k) \geq m(H, k)$  (resp.  $m(G, k) > m(H, k)$ ).

**Lemma 2.4.** ([11]) Suppose that  $G$  is a connected graph and  $T$  is an induced subgraph of  $G$  such that  $T$  is a tree and  $T$  is connected to the rest of  $G$  only by a cut vertex  $v$ . If  $T$  is replaced by a star of the same order, centered at  $v$ , then the matching energy decreases (unless  $T$  is already such a star). If  $T$  is replaced by a path, with one end at  $v$ , then the matching energy increases (unless  $T$  is already such a path).

**Lemma 2.5.** Let  $H, X, Y$  be three connected graphs disjoint in pair. Suppose that  $u, v$  are two vertices of  $H$ ,  $v'$  is a vertex of  $X$ ,  $u'$  is a vertex of  $Y$ . Let  $G$  be the graph obtained from  $H, X, Y$  by identifying  $v$  with  $v'$  and  $u$  with  $u'$ , respectively. Let  $G_1$  be the graph obtained from  $H, X, Y$  by identifying vertices  $v, v', u'$  and  $G_2$  be the graph obtained from  $H, X, Y$  by identifying vertices  $u, v', u'$ ; see Figure 2. Then  $ME(G_1) < ME(G)$  and  $ME(G_2) < ME(G)$ .

**Proof.** By Lemma 2.1, we have

$$m(G, k) = m(G - v, k) + \sum_{w \in N(v)} m(G - v - w, k - 1),$$

$$m(G_1, k) = m(G_1 - v, k) + \sum_{w' \in N(v)} m(G_1 - v - w', k - 1).$$

Since  $G_1 - v$  is a subgraph of  $G - v$  and  $G_1 - v - w'$  is a subgraph of  $G - v - w$ . By Lemma 2.3, we have  $m(G_1 - v, k) < m(G - v, k)$  and  $m(G_1 - v - w', k - 1) < m(G - v - w, k - 1)$ . Hence,  $ME(G_1) < ME(G)$ .

Similarly, we have  $ME(G_2) < ME(G)$ . ■

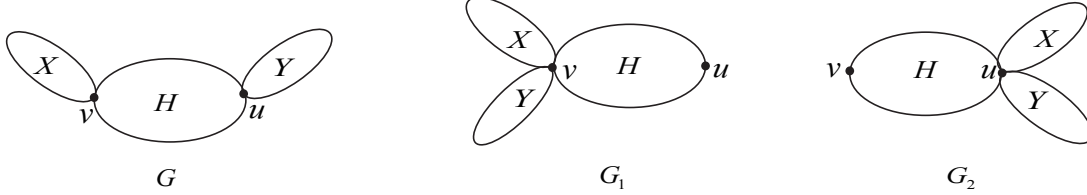


Figure 2: Graphs  $G$ ,  $G_1$  and  $G_2$ .

### 3 Main Results

**Lemma 3.1.** *For positive integers  $a$ ,  $b$  and  $t$  with  $a, b \geq 3$ ,  $ME(B_{n,a,b}^{(t)}) \leq ME(B'_{n,a,b}^{(t)})$ .*

**Proof.** Let  $B_1 = B_{n-t,a,b}^{(0)}$ . By Lemma 2.1, we have

$$\begin{aligned}
m(B_{n,a,b}^{(t)}, k) &= m(B_{n,a,b}^{(t)} - v_1, k) + m(B_{n,a,b}^{(t)} - v_1 - v, k - 1) \\
&= m(B_{n-1,a,b}^{(t-1)}, k) + m(P_{a-1} \cup P_{b-1}, k - 1) \\
&= m(B_{n-2,a,b}^{(t-2)}, k) + 2m(P_{a-1} \cup P_{b-1}, k - 1) \\
&= \dots \\
&= m(B_{n-t,a,b}^{(0)}, k) + tm(P_{a-1} \cup P_{b-1}, k - 1) \\
&= m(B_{n-t,a,b}^{(0)} - v, k) + \sum_{w \in N_{B_1}(v)} m(B_{n-t,a,b}^{(0)} - v - w, k - 1) + tm(P_{a-1} \cup P_{b-1}, k - 1) \\
&= m(P_{a-1} \cup P_{b-1}, k) + 2m(P_{a-2} \cup P_{b-1}, k - 1) \\
&\quad + 2m(P_{a-1} \cup P_{b-2}, k - 1) + tm(P_{a-1} \cup P_{b-1}, k - 1).
\end{aligned}$$

Let  $Q = B_{n,a,b}'^{(t)} - \{u, v_1, \dots, v_t\}$ . For two vertices  $u, v \in V(C_a)$ ,  $P_x$  denotes the subpath of  $C_a$  from  $u$  to  $v$  and  $P_y$  denotes the subpath from  $v$  to  $u$  in the reversed direction of  $C_a$  and  $a = x + y - 2$ . Let  $B_2 = B_{n-t,a,b}'^{(0)}$ . By Lemma 2.1, we have

$$\begin{aligned}
m(B_{n,a,b}'^{(t)}, k) &= m(B_{n,a,b}'^{(t)} - v_1, k) + m(B_{n,a,b}'^{(t)} - v_1 - u, k - 1) \\
&= m(B_{n-1,a,b}'^{(t-1)}, k) + m(Q, k - 1) \\
&= m(B_{n-2,a,b}'^{(t-2)}, k) + m(Q, k - 1) \\
&= \dots \\
&= m(B_{n-t,a,b}'^{(0)}, k) + tm(Q, k - 1) \\
&= m(B_{n-t,a,b}'^{(0)} - v, k) + \sum_{r \in N_{B_2}(v)} m(B_{n-t,a,b}'^{(0)} - v - r, k - 1) + tm(Q, k - 1)
\end{aligned}$$

$$\begin{aligned}
&= m(P_{a-1} \cup P_{b-1}, k) + 2m(P_{a-2} \cup P_{b-1}, k-1) \\
&\quad + 2m(P_{a-1} \cup P_{b-2}, k-1) + tm(Q, k-1),
\end{aligned}$$

where

$$\begin{aligned}
m(Q, k-1) &= m(Q-v, k-1) + \sum_{w' \in N_Q(v)} m(Q-v-w', k-2) \\
&= m(P_{x-2} \cup P_{y-2} \cup P_{b-1}, k-1) + m(P_{x-3} \cup P_{y-2} \cup P_{b-1}, k-2) \\
&\quad + m(P_{x-2} \cup P_{y-3} \cup P_{b-1}, k-2) + 2m(P_{x-2} \cup P_{y-2} \cup P_{b-2}, k-2) \\
&= m(P_{x-1} \cup P_{y-2} \cup P_{b-1}, k-1) + m(P_{x-2} \cup P_{y-3} \cup P_{b-1}, k-2) \\
&\quad + 2m(P_{x-2} \cup P_{y-2} \cup P_{b-2}, k-2).
\end{aligned}$$

Note that  $a = x + y - 2$ , by Lemma 2.1 we have

$$m(P_{a-1} \cup P_{b-1}, k-1) = m(P_{x-1} \cup P_{y-2} \cup P_{b-1}, k-1) + m(P_{x-2} \cup P_{y-3} \cup P_{b-1}, k-2).$$

Combining with above results,

$$m(B_{n,a,b}^{(t)}, k) - m(B_{n,a,b}^{(t)}, k) = 2tm(P_{x-2} \cup P_{y-2} \cup P_{b-2}, k-2) \geq 0.$$

Hence,  $ME(B_{n,a,b}^{(t)}) \leq ME(B_{n,a,b}^{(t)})$ . ■

Denoted by  $T(x, y, c)$  the tree with exactly one vertex  $v$  of degree 3, and  $T(x, y, c) - v = P(x-1) \cup P(y-1) \cup P(c-1)$ ; see Figure 3.

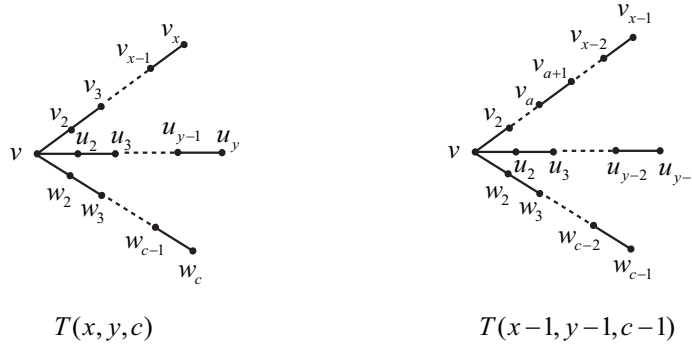


Figure 3: Trees  $T(x, y, c)$  and  $T(x-1, y-1, c-1)$ .

**Lemma 3.2.** For positive integers  $x, y, c$  and  $t$  with  $x \geq 3$ ,  $yc \geq 6$ ,  $ME(B_{n,x,y,c}^{(t)}) \leq ME(B_{n,x,y,c}^{(t)})$ .

**Proof.** Let  $T = B_{n,x,y,c}^{(t)} - \{v, v_1, \dots, v_t\} = T(x-1, y-1, c-1)$  and  $B_3 = B_{n-t,x,y,c}^{(0)}$ . By Lemma 2.1, we have

$$\begin{aligned}
m(B_{n,x,y,c}^{(t)}, k) &= m(B_{n,x,y,c}^{(t)} - v_1, k) + m(B_{n,x,y,c}^{(t)} - v_1 - v, k-1) \\
&= m(B_{n-1,x,y,c}^{(t-1)}, k) + m(T, k-1) \\
&= m(B_{n-2,x,y,c}^{(t-2)}, k) + 2m(T, k-1) \\
&= \dots \\
&= m(B_{n-t,x,y,c}^{(0)}, k) + tm(T, k-1)
\end{aligned}$$

$$\begin{aligned}
&= m(B_{n-t,x,y,c}^{(0)} - v, k) + \sum_{r \in N_{B_3}(v)} m(B_{n-t,x,y,c}^{(0)} - v - r, k - 1) + tm(T, k - 1) \\
&= m(T, k) + \sum_{r \in N_{B_3}(v)} m(T - r, k - 1) + tm(T, k - 1).
\end{aligned}$$

where

$$\begin{aligned}
m(T, k) &= m(T - u, k) + \sum_{r' \in N_T(u)} m(T - u - r', k - 1) \\
&= m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k) + m(P_{x-3} \cup P_{y-2} \cup P_{c-2}, k - 1) \\
&\quad + m(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k - 1) + m(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k - 1) \\
&= m(P_{x-1} \cup P_{y-2} \cup P_{c-2}, k) + m(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k - 1) \\
&\quad + m(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k - 1).
\end{aligned}$$

Furthermore, let  $N_{B_3}(v) = \{r_1, r_2, r_3\}$ . By Lemma 2.1, we have

$$\begin{aligned}
m(T - r_1, k - 1) &= m(T - r_1 - u, k - 1) + \sum_{r' \in N_{T-r_1}(u)} m(T - r_1 - u - r', k - 2) \\
&= m(P_{x-3} \cup P_{y-2} \cup P_{c-2}, k - 1) + m(P_{x-4} \cup P_{y-2} \cup P_{c-2}, k - 2) \\
&\quad + m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k - 2) + m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k - 2) \\
&= m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k - 1) + m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k - 2) \\
&\quad + m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k - 2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
m(T - r_2, k - 1) &= m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k - 1) + m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k - 2) \\
&\quad + m(P_{x-2} \cup P_{y-3} \cup P_{c-3}, k - 2).
\end{aligned}$$

$$\begin{aligned}
m(T - r_3, k - 1) &= m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k - 1) + m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k - 2) \\
&\quad + m(P_{x-2} \cup P_{y-3} \cup P_{c-3}, k - 2).
\end{aligned}$$

Combining with above results,

$$\begin{aligned}
m(B_{n,x,y,c}^{(t)}, k) &= m(P_{x-1} \cup P_{y-2} \cup P_{c-2}, k) + m(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k - 1) \\
&\quad + m(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k - 1) + 3m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k - 1) \\
&\quad + 2m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k - 2) + 2m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k - 2) \\
&\quad + 2m(P_{x-2} \cup P_{y-3} \cup P_{c-3}, k - 2) + tm(P_{x-1} \cup P_{y-2} \cup P_{c-2}, k - 1) \\
&\quad + tm(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k - 2) + tm(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k - 2).
\end{aligned}$$

Let  $H = B_{n,x,y,c}'^{(t)} - \{w, v_1, \dots, v_t\}$ . Similarly, we have

$$\begin{aligned}
m(B_{n,x,y,c}'^{(t)}, k) &= m(P_{x-1} \cup P_{y-2} \cup P_{c-2}, k) + m(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k - 1) \\
&\quad + m(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k - 1) + 3m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k - 1) \\
&\quad + 2m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k - 2) + 2m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k - 2) \\
&\quad + 2m(P_{x-2} \cup P_{y-3} \cup P_{c-3}, k - 2) + tm(H, k - 1).
\end{aligned}$$

Therefore,  $m(B'_{n,x,y,c}^{(t)}, k) - m(B_{n,x,y,c}^{(t)}, k) = tm(H, k-1) - tm(T, k-1)$ . Let  $H - v = T(a, y-1, c-1) \cup P_{x-a-2}$ .

By lemma 2.1, we have

$$\begin{aligned}
m(H, k-1) &= m(H-v, k-1) + \sum_{w' \in N_H(v)} m(H-v-w', k-2) \\
&= m(T(a, y-1, c-1) \cup P_{x-a-2}, k-1) + m(T(a, y-1, c-1) \cup P_{x-a-3}, k-2) \\
&\quad + m(T(a, y-2, c-1) \cup P_{x-a-2}, k-2) + m(T(a, y-1, c-2) \cup P_{x-a-2}, k-2) \\
&= m(T(a, y-1, c-1) \cup P_{x-a-1}, k-1) + m(T(a, y-2, c-1) \cup P_{x-a-2}, k-2) \\
&\quad + m(T(a, y-1, c-2) \cup P_{x-a-2}, k-2).
\end{aligned}$$

Let  $e = v_a v_{a+1} \in T$ ; see Figure 3. Then

$$\begin{aligned}
m(T, k-1) &= m(T-e, k-1) + m(T-u'-v', k-2) \\
&= m(T(a, y-1, c-1) \cup P_{x-a-1}, k-1) + m(T(a-1, y-1, c-1) \cup P_{x-a-2}, k-2).
\end{aligned}$$

Hence,

$$\begin{aligned}
&m(H, k-1) - m(T, k-1) \\
&= m(T(a, y-2, c-1) \cup P_{x-a-2}, k-2) + m(T(a, y-1, c-2) \cup P_{x-a-2}, k-2) \\
&\quad - m(T(a-1, y-1, c-1) \cup P_{x-a-2}, k-2) \\
&= m(P_{c-2} \cup P_{a+y-3} \cup P_{x-a-2}, k-2) + m(P_{c-3} \cup P_{a-1} \cup P_{y-3} \cup P_{x-a-2}, k-3) \\
&\quad + m(T(a, y-1, c-2) \cup P_{x-a-2}, k-2) \\
&\quad - m(P_{c-2} \cup P_{a+y-3} \cup P_{x-a-2}, k-2) - m(P_{c-3} \cup P_{a-2} \cup P_{y-2} \cup P_{x-a-2}, k-3) \\
&= m(P_{c-3} \cup P_{a-1} \cup P_{y-3} \cup P_{x-a-2}, k-3) + m(P_{c-3} \cup P_{a+y-2} \cup P_{x-a-2}, k-2) \\
&\quad + m(P_{c-4} \cup P_{a-1} \cup P_{y-2} \cup P_{x-a-2}, k-3) - m(P_{c-3} \cup P_{a-2} \cup P_{y-2} \cup P_{x-a-2}, k-3) \\
&= m(P_{c-3} \cup P_{a-1} \cup P_{y-3} \cup P_{x-a-2}, k-3) + m(P_{c-3} \cup P_{a-1} \cup P_{y-1} \cup P_{x-a-2}, k-2) \\
&\quad + m(P_{c-4} \cup P_{a-1} \cup P_{y-2} \cup P_{x-a-2}, k-3).
\end{aligned}$$

By Lemmas 2.2 and 2.3, we have  $m(H, k-1) - m(T, k-1) \geq 0$ . That is to say  $B'_{n,x,y,c}^{(t)}, k) - m(B_{n,x,y,c}^{(t)}, k) \geq 0$ .

Thus, for positive integers  $x, y, c$  and  $t$  with  $x \geq 3, yc \geq 6$ ,  $ME(B_{n,x,y,c}^{(t)}) \leq ME(B'_{n,x,y,c}^{(t)})$ . ■

**Lemma 3.3.** *Let  $G \in \mathcal{B}(n)$ . Then*

(i) *If  $G$  contains exactly two cycles, say  $C_a$  and  $C_b$ , then  $ME(G) \geq ME(B_{n,a,b}^{(t)})$ , the equality holds if and only if  $G \cong B_{n,a,b}^{(t)}$ .*

(ii) *If  $G$  contains exactly three cycles, say  $B_{x,y,c}$ , then  $ME(G) \geq ME(B_{n,x,y,c}^{(t)})$ , the equality holds if and only if  $G \cong B_{n,x,y,c}^{(t)}$ .*

**Proof.** (i) If  $G$  contains exactly two cycles, say  $C_a$  and  $C_b$ . Assume that  $C_a$  connects  $C_b$  by a path  $P_{l+2}$  with  $l \geq -1$ . It is easy to know that  $C_a$  and  $C_b$  have exactly one vertex in common when  $l = -1$ . Let  $C_a = u_1 u_2 \dots u_a u_1$ ,  $C_b = v_1 v_2 \dots v_b v_1$  and  $P_{l+2} = u_1 w_1 w_2 \dots w_l v_1$ . Set  $V^*(G) = \{u_i : d(u_i) \geq 3, 2 \leq i \leq a\} \cup \{v_i : d(v_i) \geq 3, 2 \leq i \leq b\} \cup \{w_i : d(w_i) \geq 3, 1 \leq i \leq$

$l\} \cup \{u_1 : d(u_1) \geq 4\} \cup \{v_1 : d(v_1) \geq 4\}$ . Suppose that  $|V^*(G)| = k$ , and relabel the vertices in  $V^*(G)$  as  $\{r_1, r_2, \dots, r_k\}$ .

Let  $r_i \in V^*(G)$  and  $T_i$  be a subtree of  $G - E(C_a \cup C_b \cup P_{l+2})$  which contains  $r_i$  and  $|V(T_i)| = p_i + 1$ . Let  $V'(T_i) = V(T_i) - r_i$ . Denote  $H = G - V'(T_i)$ .

Then  $G = Hr_iT_i$ . By Lemma 2.4, we have  $ME(Hr_iT_i) \geq ME(Hr_iS_{p_i+1})$ . Thus repeatedly using Lemma 2.4, we have  $ME(G) \geq ME(B_n(p_1, p_2, \dots, p_k))$ , where  $B_n(p_1, p_2, \dots, p_k)$  is a bicyclic graph with  $n$  vertices created from  $C_a u_1 P_{l+2} v_1 C_b$  by attaching  $p_i$  pendent vertices to  $r_i \in V^*(G)$ ,  $1 \leq i \leq k$ , respectively.

Let  $X = S_{p_i+1}$ ,  $Y = S_{p_j+1}$ ,  $V'(X) = V(X) - r_i$  and  $V'(Y) = V(Y) - r_j$ . Denote  $H' = G - V'(X) - V'(Y)$ . Then  $B_n(p_1, p_2, \dots, p_k) = Xr_iH'r_jY$ , where  $r_i$  and  $r_j$  are centers of  $X$  and  $Y$ , respectively. By Lemma 2.5, we have

$$ME(G) \geq ME(B_n(p_1, \dots, p_i, \dots, p_j, \dots, p_k)) > ME(B_n(p_1, \dots, p_{i+j}, \dots, 0, \dots, p_k)),$$

or

$$ME(G) \geq ME(B_n(p_1, \dots, p_i, \dots, p_j, \dots, p_k)) > ME(B_n(p_1, \dots, 0, \dots, p_{i+j}, \dots, p_k)).$$

Repeatedly using above step, and by Lemmas 2.1, 2.2 and 2.3, we obtain either

$$ME(G) \geq ME(B_n(p_1, \dots, p_i, \dots, p_j, \dots, p_k)) > \dots > ME(B_{n,a,b}^{(t)}),$$

or

$$ME(G) \geq ME(B_n(p_1, \dots, p_i, \dots, p_j, \dots, p_k)) > \dots > ME(B_{n,a,b}'^{(t)}).$$

Combining with Lemma 3.1, if  $G$  contains exactly two cycles, then we have  $ME(G) \geq ME(B_{n,a,b}^{(t)})$ , the equality holds if and only if  $G \cong B_{n,a,b}^{(t)}$ .

(ii) If  $G$  contains exactly three cycles, say  $B_{x,y,c}$ . Let  $P_x = u_1 u_2 \dots u_x$ ,  $P_y = u_1 v_2 \dots v_{y-1} u_x$  and  $P_c = u_1 w_2 \dots w_{c-1} u_x$ . Set  $V^*(G) = \{u_i : d(u_i) \geq 3, 2 \leq i \leq x-1\} \cup \{v_i : d(v_i) \geq 3, 2 \leq i \leq y-1\} \cup \{w_i : d(w_i) \geq 3, 2 \leq i \leq c-1\} \cup \{u_1 : d(u_1) \geq 4\} \cup \{u_x : d(u_x) \geq 4\}$ . Assume that  $|V^*(G)| = k$ , and relabel the vertices in  $V^*(G)$  as  $\{r_1, r_2, \dots, r_k\}$ .

Let  $r_i \in V^*(G)$  and  $T_i$  be a subtree of  $G - E(B_{x,y,c})$  which contains  $r_i$  and  $|V(T_i)| = p_i + 1$ . Let  $V'(T_i) = V(T_i) - r_i$ . Denote  $H = G - V'(T_i)$ .

Then  $G = Hr_iT_i$ . By Lemma 2.4, we have  $ME(Hr_iT_i) \geq ME(Hr_iS_{p_i+1})$ . Thus repeated using Lemma 2.4, we have  $ME(G) \geq ME(B'_n(p_1, p_2, \dots, p_k))$ , where  $B'_n(p_1, p_2, \dots, p_k)$  is a bicyclic graph with  $n$  vertices created from  $B_{x,y,c}$  by attaching  $p_i$  pendent vertices to  $r_i \in V^*(G)$ ,  $1 \leq i \leq k$ , respectively.

Let  $X = S_{p_i+1}$ ,  $Y = S_{p_j+1}$ ,  $V'(X) = V(X) - r_i$  and  $V'(Y) = V(Y) - r_j$ . Denote  $H' = G - V'(X) - V'(Y)$ . Then  $B'_n(p_1, p_2, \dots, p_k) = Xr_iH'r_jY$ , where  $r_i$  and  $r_j$  are centers of  $X$  and  $Y$ , respectively. By Lemma 2.5, we have

$$ME(G) \geq ME(B'_n(p_1, \dots, p_i, \dots, p_j, \dots, p_k)) > ME(B'_n(p_1, \dots, p_{i+j}, \dots, 0, \dots, p_k)),$$

or

$$ME(G) \geq ME(B'_n(p_1, \dots, p_i, \dots, p_j, \dots, p_k)) > ME(B'_n(p_1, \dots, 0, \dots, p_{i+j}, \dots, p_k)).$$

Repeatedly using above step, we obtain either

$$ME(G) \geq ME(B'_n(p_1, \dots, p_i, \dots, p_j, \dots, p_k)) > \dots > ME(B_{n,x,y,c}'^{(t)}),$$



or

$$ME(G) \geq ME(B'_n(p_1, \dots, p_i, \dots, p_j, \dots, p_k)) > \dots > ME(B'^{(t)}_{n,x,y,c}).$$

Combining with Lemma 3.2, if  $G$  contains exactly three cycles, then we have  $ME(G) \geq ME(B^{(t)}_{n,x,y,c})$ , the equality holds if and only if  $G \cong B^{(t)}_{n,x,y,c}$ .  $\blacksquare$

**Theorem 3.4.** For positive integers  $a, b$  and  $t$  with  $a \geq 4, b \geq 3$ ,  $ME(B^{(t+1)}_{n,a-1,b}) < ME(B^{(t)}_{n,a,b})$ .

**Proof.** By the proof of Lemma 3.1, we have

$$\begin{aligned} m(B^{(t+1)}_{n,a-1,b}, k) - m(B^{(t)}_{n,a,b}, k) &= m(P_{a-2} \cup P_{b-1}, k) + 2m(P_{a-3} \cup P_{b-1}, k-1) \\ &\quad + 2m(P_{a-2} \cup P_{b-2}, k-1) + (t+1)m(P_{a-2} \cup P_{b-1}, k-1) \\ &\quad - m(P_{a-1} \cup P_{b-1}, k) - 2m(P_{a-2} \cup P_{b-1}, k-1) \\ &\quad - 2m(P_{a-1} \cup P_{b-2}, k-1) - tm(P_{a-1} \cup P_{b-1}, k-1) \\ &= m(P_{a-2} \cup P_{b-1}, k) + 2m(P_{a-3} \cup P_{b-1}, k-1) \\ &\quad + 2m(P_{a-2} \cup P_{b-2}, k-1) + (t+1)m(P_{a-2} \cup P_{b-1}, k-1) \\ &\quad - m(P_{a-2} \cup P_{b-1}, k) - m(P_{a-2} \cup P_{b-1}, k-1) \\ &\quad - m(P_{a-3} \cup P_{b-1}, k-1) - m(P_{a-2} \cup P_{b-1}, k-1) \\ &\quad - 2m(P_{a-1} \cup P_{b-2}, k-1) - tm(P_{a-1} \cup P_{b-1}, k-1) \\ &= m(P_{a-3} \cup P_{b-1}, k-1) - m(P_{a-2} \cup P_{b-1}, k-1) \\ &\quad + 2m(P_{a-2} \cup P_{b-2}, k-1) - 2m(P_{a-1} \cup P_{b-2}, k-1) \\ &\quad + tm(P_{a-2} \cup P_{b-1}, k-1) - tm(P_{a-1} \cup P_{b-1}, k-1). \end{aligned}$$

By Lemmas 2.2 and 2.3, we have  $m(B^{(t+1)}_{n,a-1,b}, k) - m(B^{(t)}_{n,a,b}, k) < 0$ . So  $ME(B^{(t+1)}_{n,a-1,b}) < ME(B^{(t)}_{n,a,b})$ .  $\blacksquare$

**Theorem 3.5.** For positive integers  $x, y, c$  and  $t$  with  $x \geq 4, y, c \geq 2, yc \geq 6$ ,  $ME(B^{(t+1)}_{n,x-1,y,c}) < ME(B^{(t)}_{n,x,y,c})$ .

**Proof.** By the proof of Lemma 3.2, we have

$$\begin{aligned} m(B^{(t+1)}_{n,x-1,y,c}, k) - m(B^{(t)}_{n,x,y,c}, k) &= m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k) + m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-1) \\ &\quad + m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-1) + 3m(P_{x-3} \cup P_{y-2} \cup P_{c-2}, k-1) \\ &\quad + 2m(P_{x-4} \cup P_{y-2} \cup P_{c-3}, k-2) + 2m(P_{x-4} \cup P_{y-3} \cup P_{c-2}, k-2) \\ &\quad + 2m(P_{x-3} \cup P_{y-3} \cup P_{c-3}, k-2) + (t+1)m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k-1) \\ &\quad + (t+1)m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-2) + (t+1)m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-2) \\ &\quad - m(P_{x-1} \cup P_{y-2} \cup P_{c-2}, k) - 3m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k-1) \\ &\quad - m(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k-1) - 2m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-2) \\ &\quad - m(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k-1) - 2m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-2) \\ &\quad - 2m(P_{x-2} \cup P_{y-3} \cup P_{c-3}, k-2) - tm(P_{x-1} \cup P_{y-2} \cup P_{c-2}, k-1) \\ &\quad - tm(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k-2) - tm(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k-2) \end{aligned}$$

$$\begin{aligned}
&= m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k) + m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-1) \\
&\quad + m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-1) + 3m(P_{x-3} \cup P_{y-2} \cup P_{c-2}, k-1) \\
&\quad + 2m(P_{x-4} \cup P_{y-2} \cup P_{c-3}, k-2) + 2m(P_{x-4} \cup P_{y-3} \cup P_{c-2}, k-2) \\
&\quad + 2m(P_{x-3} \cup P_{y-3} \cup P_{c-3}, k-2) + (t+1)m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k-1) \\
&\quad + (t+1)m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-2) + (t+1)m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-2) \\
&\quad - [m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k) + m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k-1) \\
&\quad + 2m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k-1) + m(P_{x-3} \cup P_{y-2} \cup P_{c-2}, k-1)] \\
&\quad - [m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-1) + m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-2) \\
&\quad + m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-2) + m(P_{x-4} \cup P_{y-3} \cup P_{c-2}, k-2)] \\
&\quad - [m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-1) + m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-2) \\
&\quad + m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-2) + m(P_{x-4} \cup P_{y-2} \cup P_{c-3}, k-2)] \\
&\quad - 2m(P_{x-2} \cup P_{y-3} \cup P_{c-3}, k-2) - tm(P_{x-1} \cup P_{y-2} \cup P_{c-2}, k-1) \\
&\quad - tm(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k-2) - tm(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k-2) \\
&= 2m(P_{x-3} \cup P_{y-2} \cup P_{c-2}, k-1) - 2m(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k-1) \\
&\quad + m(P_{x-4} \cup P_{y-2} \cup P_{c-3}, k-2) - m(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-2) \\
&\quad + m(P_{x-4} \cup P_{y-3} \cup P_{c-2}, k-2) - m(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-2) \\
&\quad + 2m(P_{x-3} \cup P_{y-3} \cup P_{c-3}, k-2) - 2m(P_{x-2} \cup P_{y-3} \cup P_{c-3}, k-2) \\
&\quad + tm(P_{x-2} \cup P_{y-2} \cup P_{c-2}, k-1) - tm(P_{x-1} \cup P_{y-2} \cup P_{c-2}, k-1) \\
&\quad + tm(P_{x-3} \cup P_{y-3} \cup P_{c-2}, k-2) - tm(P_{x-2} \cup P_{y-3} \cup P_{c-2}, k-2) \\
&\quad + tm(P_{x-3} \cup P_{y-2} \cup P_{c-3}, k-2) - tm(P_{x-2} \cup P_{y-2} \cup P_{c-3}, k-2).
\end{aligned}$$

By Lemmas 2.2 and 2.3, we have  $m(B_{n,x-1,y,c}^{(t+1)}, k) - m(B_{n,x,y,c}^{(t)}, k) < 0$ . So  $ME(B_{n,x-1,y,c}^{(t+1)}) < ME(B_{n,x,y,c}^{(t)})$ .  $\blacksquare$

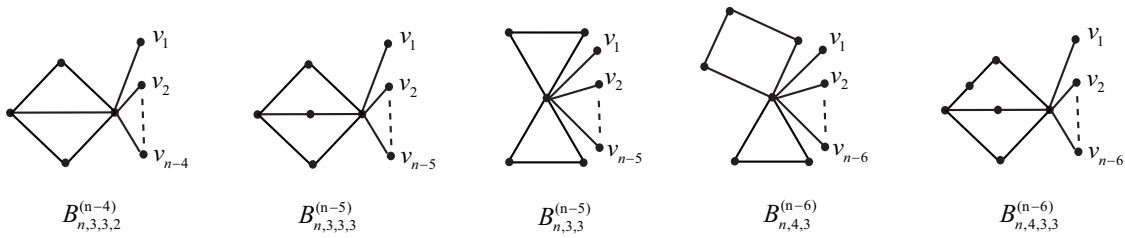


Figure 4: Bicyclic graphs  $B_{n,3,3,2}^{(n-4)}$ ,  $B_{n,3,3,3}^{(n-5)}$ ,  $B_{n,3,3}^{(n-5)}$ ,  $B_{n,4,3}^{(n-6)}$  and  $B_{n,4,3,3}^{(n-6)}$ .

**Theorem 3.6.** Let  $G \in B_n$ . For  $n > 5$ , if  $G$  is not isomorphic to any member in  $\{B_{n,3,3,2}^{(n-4)}, B_{n,3,3,3}^{(n-5)}, B_{n,3,3}^{(n-5)}, B_{n,4,3}^{(n-6)}, B_{n,4,3,3}^{(n-6)}\}$ ; see Figure 4, then we have

$$ME(G) > ME(B_{n,4,3,3}^{(n-6)}) > ME(B_{n,4,3}^{(n-6)}) > ME(B_{n,3,3}^{(n-5)}) > ME(B_{n,3,3,3}^{(n-5)}) > ME(B_{n,3,3,2}^{(n-4)}).$$

**Proof.** Note that

$$m(B_{n,3,3,2}^{(n-4)}, 1) = n + 1, m(B_{n,3,3,2}^{(n-4)}, 2) = 2n - 6, m(B_{n,3,3,2}^{(n-4)}, k) = 0 \text{ for } k \geq 3;$$

$$\begin{aligned}
m(B_{n,3,3,3}^{(n-5)}, 1) &= n+1, m(B_{n,3,3,3}^{(n-5)}, 2) = 3n-9, m(B_{n,3,3,3}^{(n-5)}, k) = 0 \text{ for } k \geq 3; \\
m(B_{n,3,3}^{(n-5)}, 1) &= n+1, m(B_{n,3,3}^{(n-5)}, 2) = 2n-5, m(B_{n,3,3}^{(n-5)}, 3) = n-5, \\
m(B_{n,3,3}^{(n-5)}, k) &= 0 \text{ for } k \geq 4; \\
m(B_{n,4,3}^{(n-6)}, 1) &= n+1, m(B_{n,4,3}^{(n-6)}, 2) = 3n-8, m(B_{n,4,3}^{(n-6)}, 3) = 2n-10, \\
m(B_{n,4,3}^{(n-6)}, k) &= 0 \text{ for } k \geq 4; \\
m(B_{n,4,3,3}^{(n-6)}, 1) &= n+1, m(B_{n,4,3,3}^{(n-6)}, 2) = 4n-13, m(B_{n,4,3,3}^{(n-6)}, 3) = 2n-10, \\
m(B_{n,4,3,3}^{(n-6)}, k) &= 0 \text{ for } k \geq 4.
\end{aligned}$$

For  $n > 5$ ,  $m(B_{n,3,3,3}^{(n-5)}, 2) - m(B_{n,3,3,2}^{(n-4)}, 2) = n-3 > 0$ . Thus  $ME(B_{n,3,3,3}^{(n-5)}) > ME(B_{n,3,3,2}^{(n-4)})$ . Similarly, we have  $ME(B_{n,4,3,3}^{(n-6)}) > ME(B_{n,4,3}^{(n-6)}) > ME(B_{n,3,3}^{(n-5)})$ . In addition, we have

$$\begin{aligned}
\alpha(B_{n,3,3,3}^{(n-5)}, x) &= x^n - (n+1)x^{n-2} + (3n-9)x^{n-4}; \\
\alpha(B_{n,3,3}^{(n-5)}, x) &= x^n - (n+1)x^{n-2} + (2n-5)x^{n-4} - (n-5)x^{n-6}.
\end{aligned}$$

By the definition of matching energy, we have

$$\begin{aligned}
ME(B_{n,3,3,3}^{(n-5)}) &= 2\sqrt{\frac{n+1 + \sqrt{(n+1)^2 - 4(3n-9)}}{2}} + 2\sqrt{\frac{n+1 - \sqrt{(n+1)^2 - 4(3n-9)}}{2}}; \\
ME(B_{n,3,3}^{(n-5)}) &= 2 + 2\sqrt{\frac{n + \sqrt{n^2 - 4(n-5)}}{2}} + 2\sqrt{\frac{n - \sqrt{n^2 - 4(n-5)}}{2}}.
\end{aligned}$$

By a directly calculation, we have  $ME(B_{n,3,3}^{(n-5)}) > ME(B_{n,3,3,3}^{(n-5)})$ .

Hence,  $ME(B_{n,4,3,3}^{(n-6)}) > ME(B_{n,4,3}^{(n-6)}) > ME(B_{n,3,3}^{(n-5)}) > ME(B_{n,3,3,3}^{(n-5)}) > ME(B_{n,3,3,2}^{(n-4)})$ . ■

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